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ON EQUALITY OF SPECTRA AND ESSENTIAL SPECRA OF OPERATORS IN HILBERT SPACES

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ABSTRACT:

It is a well known fact in operator theory that for any operator A, the essential spectrum of A is contained in the spectrum of A. The task of finding conditions under which the two are equal has been the subject matter in a number of several research papers. In this paper we show that quasisimilar pure dominant operators have their essential spectra equal to their spectra provided one of the intertwining quasiaffinities is compact.

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1. INTRODUCTION

Let H be a complex Hilbert space and B(H) denote the Banach algebra of all bounded linear operators on H. An operator $A \in B(H)$ is said to be a quasiaffinity if A is both one-one and has dense range. Two operators A and B are said to be similar if there is an invertible operator S such that $AS = SB$, while A and B are said to be quasisimilar if there exist quasiaffinities X and Y such that $AX = XB$ and $BY = YA$.

The concept of quasisimilarity and equality of spectra has been considered by a number of authors among them S. Clary [1] who showed that quasisimilar hyponormal operators have equal spectra. J.M. Khalagai and B. Nyamai [2] showed that if A and B are quasisimilar with A dominant and B M-hyponormal them B have equal spectra. B.P. Duggal [3] Showed that if $A_{i,}$ $i = 1,2$ are quasi-similar $P -$ hyponormal such that $\cup_{i,}$ $i = 1,2$ is unitary in the polar decomposition $A_{i,} = \cup_i |A_i|$ then A_1 and A_2 have not only same spectra but also same essential spectra.

The Problem of looking for conditions under which the essential spectrum is equal to spectrum of a given operator has also been considered by a number of authors. In particular J.P. Williams [5] apart from showing that there are several cases under which quasisimilar operators A and B have equal essential spectra also proved the following result on equality of spectrum and essential spectrum of a given operator.

Theorem A [5]

Suppose that T is a pure dominant operator, K is a compact operator having dense range and $KT = TK$. Then spectrum of T is equal to essential spectrum of T.

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It is at this point that we also pick up the quest of delving into this theory in this paper.

Definition1

Let $x \in \mathcal{H}$ We define $\varrho_T(x)$ to be the set of complex numbers α for which there exists a neighbourhood V_α of α with u analytic on having values in H such that $(ZI - T)u(z) = x$ on V_α.

This set is open and contains the complement of the spectrum $\sigma(T)$ of T. The function u is called a local resolvent of T on V_α. By definition the local spectrum of T at x denoted by $\sigma(T, x)$ is the complement of $\varrho_T(x)$ and so is a compact subset of $\sigma(T)$. We say that T has the single valued extension property (in short SVEP) if $(zI - T)\, u(z) = 0$ implies $u = 0$ for any analytic function u defined on any domain D of a complex plane with values in H.

An operator $T \in H$ is said to satisfy Dunford's property (C) if for each closed subset F of the complex plane the corresponding local spectrum subspace $H_T(F) - \{x \in H: \sigma(T, x) \subset F\}$ is closed.

Theorem B [5]

Suppose A and B are dominant operators satisfying Dunford's property (C) and are quasisimilar with at least one of the implementing quasiaffinities compact, then A and B have equal spectra and also equal essential spectra.

2. NOTATION AND TERMINOLOGY

Given an operator $A \in B(H)$the spectrum of A is denoted by $\sigma(A)$.

Thus $\sigma(A) = \left\{\lambda \in \mathbb{C} \;: A - \lambda\, I \text{ is not invertible}\right\}$ where $\mathbb{C}$ is the complex number field. The commutator of A and B is denoted by $[A, B]$ where $[A, B] = AB - BA$.

We denote range of A and Kernel of A by ran A and Ker A. An operator A is said to be dominant if to each $\lambda \in \mathbb{C}$ there corresponds a number $M_\lambda \geq 1$ such that $\left\|(A - \lambda)^* x\right\| \leq M\|(\Lambda - \lambda)x\|\ \forall\, x \in H$. But A is said to be pure dominant if there exists no non-trivial reducing subspace of A on which the restriction of A is normal. A is M −hyponormal if $\exists\, M > M_\lambda$ for all λ in the definition of dominant operator such that $\left\|(A - \lambda)^* x\right\| \leq M\|(A - \lambda)x\|\ \ \forall\, x \in H$.

Also A is:

Hyponormal if $A^*A \geq AA^*$

Quasinormal if $[A^*A, A] = 0$

P −hyponormal if for $\cup < p \leq 1, (A^*A)^p \geq (AA^*)^p$

Normal if $[A, A^*] = 0$

Self adjoint if $A = A^*$

Partial isometry if $A = A\,A^*A$

Isometry $A^*A = I$

Unitary if $A^*A = AA^* = I$

Fredholm if ran$A$ is closed and both Ker$A$ and Ker A^* are finite dimensional.

The essential spectrum of A is denoted by $\sigma_e(A)$. Thus $\sigma_e(A) = \{\lambda \in \mathbb{C}: A - \lambda I$ is not fredholm$\}$

An operator A is said to be compact if it maps the unit ball into a compact set. Equivalently A is compact if and only if $\lim_{n\to\infty} Ax_n = 0$ for each sequence (x_n) in H weakly converging to 0.

Note that the following result about properties of compact operators can easily be verified.

LEMMA C [2]

(i) If A is compact then so is A^*
(ii) If A is compact and B is bounded then AB and BA are also compact.

We also have the following inclusion of classes of operators.
Normal ⊂ hyponormal ⊂ p-hyponormal
and Hyponormal ⊂M-hyponormal ⊂ Dominant.

3. RESULTS

Theorem 1

Let $A, B \in B(H)$ be quasisimilar pure dominant operators with at least one of the intertwining quasiaffinities compact. Then we have:

$\sigma_e(A) = \sigma(A)$

$\sigma_e(B) = \sigma(B)$

Proof: Since A and B are quasisimilar there exist two quasiaffinities X and Y such that

$AX = XB$ and $BY = YA$

We also have that either X or Y is compact implies XY and YX are compact operators each with dense range. It can also be verified easily that $[A, XY] = 0$ and $[B, YX] = 0$. Now from theorem A above we have $\sigma_e(A) = \sigma(A)$ and $\sigma_e(B) = \sigma(B)$

Hence the result.

Corollary 1

Let $A, B \in B(H)$ be quasiinvertible operator's with either A or B compact. If AB and BA are pure dominant operators then we have $\sigma_e(AB) = \sigma(AB)$ and $\sigma_e(BA) = \sigma(BA)$

Proof: We first note that quasi invertibility is the same as quasiaffinity. Thus A and B are quasiaffinities and AB and BA are also compact quasiaffinities which are quasisimilar since we have:

$(AB)A = A(BA)$ and

$(BA)B = B(AB)$

Hence by theorem 1 above $\sigma_e(AB) = \sigma(AB)$ and $\sigma_e(BA) = \sigma(BA)$

Corollary 2

Let A and B be quasisimilar pure dominant operators which satisfy Dunford's condition (C) with at least one of the implementing quasiaffinities compact. Then we have

$\sigma_e(A) = \sigma(A) = \sigma(B) = \sigma_e(B)$

Proof: The proof follows easily from both theorems B and theorem 1.

Corollary 3

Let $A, B \in B(H)$ be quasiinvertible operators with either A or B compact. If AB and BA are pure dominant operators satisfying Dunford's property (C) then we have

$\sigma_e(AB) = \sigma(AB) = \sigma(BA) = \sigma_e(BA)$

Remark;$Then\ its\ clear\ \sigma_e(AB) = \sigma_e(BA)$

Proof: If follows easily from both theorem B and corollary 1 above.

Theorem 2

Let A be a pure dominant operator and B be such that $AX = XB$ implies $A^*X = XB^*$ where X is a compact quasiaffinity, then $\sigma_e(A) = \sigma(A)$.

Proof: Since $AX = XB$ implies $A^*X = XB^*$ it can easily be verified that

$[A, XX^*] = 0$ and

$[B, X^*X] = 0$

Where XX^* is compact with dense range. Hence by theorem A above $\sigma_e(A) = \sigma(A)$

Corollary 4

If A is a pure dominant operator such that

$AX = XA^*$ and $A^*X = XA$

Where X is a compact quasiaffinity then $\sigma_e(A) = \sigma(A)$

Proof: In this case $[A, XX^*] = 0$ where XX^* is compact with dense range, hence the result follows from theorem 2 above.

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